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## QUANTUM OPTICS

### Contents

This text book originated out of a graduate course of lectures in Quantum Optics given at the University of Waikato and the University of Auckland. A broad range of material is covered in this book ranging from introductory concepts to current research topics. A pedagogic description of the techniques of quantum optics and their applications to physical systems is presented. Particular emphasis is given to systems where the theoretical predictions have been confirmed by experimental observation.

The material presented in this text could be covered in a two semester course. Alternatively the introductory material in Chaps. 1-6 and selected topics from the later chapters would be suitable for a one semester course. For example, for material involving the interaction of light with atoms Chaps. 10-13 would be appropriate, whereas for material on squeezed light Chaps. 7 and 8 are required. Chaps. 14-16 describe the interrelation of fundamental topics in quantum mechanics with quantum optics. The final chapter on atomic optics gives an introduction to this new and rapidly developing field.

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