χ-Oszillator

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Literatur

[001] Keine für vorliegenden Text.

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1 Schrödinger-Gleichung und Freies Teilchen

Gegeben ist die Schrödinger Gleichung in stationärer Form: [001]

$$-\frac{\bar{h}^2}{2m}\cdot\phi''+E_{\text{Pot}}\cdot\phi=E\phi$$

Es soll ein freies Teilchen vorliegen:

$$E_{\text{Pot}}=0$$

⇒

$$\frac{\bar{h}^2}{2m}\cdot\phi''+E\cdot\phi=0$$

⇒

$$\phi''+\frac{2m}{\bar{h}^2}\cdot E\cdot\phi=0$$

Lösung durch den Ansatz nach Allgemeine, homogene, lineare Differentialgleichung 2. Ordnung. Die Charakteristische Gleichung:

$$\lambda^2+\frac{2m}{\bar{h}^2}\cdot E=0$$

⇒

$$\lambda_{1;2}=\pm\sqrt{-\frac{2m}{\bar{h}^2}\cdot E}$$

⇒

$$a\pm ib=0\pm i\sqrt{\frac{2m}{\bar{h}^2}\cdot E}$$

Schwingungsfähig dann, wenn gilt:

$$\frac{2m}{\bar{h}^2}\cdot E>0$$

Ist immer erfüllt. Die Lösung lautet daher:

$$\phi=\sin(bx)$$

⇒

$$\phi=\sin\left(\frac{x}{\bar{h}}\cdot\sqrt{2m\cdot E}\right)$$

Mit:

$$b=\omega_x=\sqrt{\frac{2m}{\bar{h}^2}\cdot E}\left[\sqrt{\frac{\text{kg}}{\text{J}^2\text{s}^2}\text{J}}=\sqrt{\frac{\text{kg}}{\text{Nms}^2}}=\sqrt{\frac{\text{kgs}^2}{\text{kgm}^2\text{s}^2}}=\sqrt{\frac{1}{\text{m}^2}}=\frac{1}{\text{m}}\right]$$

⇒

$$\omega_x=\frac{\sqrt{2m\cdot E}}{\bar{h}}$$

⇒

$$f_x=\frac{1}{h}\cdot\sqrt{2m\cdot E}\left[\frac{1}{\text{Js}}\sqrt{\text{kgJ}}=\frac{1}{\text{Nms}}\sqrt{\text{kgNm}}=\frac{\text{s}^2}{\text{kgm}^2\text{s}}\sqrt{\text{kg}\frac{\text{kgm}}{\text{s}^2}\text{m}}=\frac{\text{skgm}}{\text{kgm}^2\text{s}}=\frac{1}{\text{m}}\right]$$

Kontrolle der Lösung:

$$\phi=\sin\left(\frac{x}{\bar{h}}\cdot\sqrt{2m\cdot E}\right)$$
$$\phi'=\frac{\sqrt{2m\cdot E}}{\bar{h}}\cdot\cos\left(\frac{x}{\bar{h}}\cdot\sqrt{2m\cdot E}\right)$$
$$\phi''=-\frac{2m\cdot E}{\bar{h}^2}\cdot\sin\left(\frac{x}{\bar{h}}\cdot\sqrt{2m\cdot E}\right)$$

⇒

$$\phi''+\frac{2m}{\bar{h}^2}\cdot E\cdot\phi=0$$

⇒

$$-\frac{2m}{\bar{h}^2}\cdot E\cdot\sin\left(\frac{x}{\bar{h}}\cdot\sqrt{2m\cdot E}\right)+\frac{2m}{\bar{h}^2}\cdot E\cdot\sin\left(\frac{x}{\bar{h}}\cdot\sqrt{2m\cdot E}\right)=0$$

$\Rightarrow$

$$0 = 0$$

Die Lösung ist exakt, jedoch muss gelten:

$$C \int\limits_{-\infty}^{+\infty} \phi\bar{\phi} dx = 1$$

$\Rightarrow$

$$\phi = \sin\left(\frac{x}{\bar{h}} \cdot \sqrt{2m \cdot E}\right) = \sin kx$$

In Komplexschreibweise:

$$\phi = \sin kx = \cos kx + i \sin kx$$

$\Rightarrow$

$$\phi = \cos kx + i \sin kx = e^{ikx}$$

$\Rightarrow$

$$\bar{\phi} = e^{-ikx}$$

$\Rightarrow$

$$\phi\bar{\phi} = e^{ikx} \cdot e^{-ikx} = e^0 = 1$$

$\Rightarrow$

$$C \int\limits_{0}^{L} dx = 1$$

$\Rightarrow$

$$C = \frac{1}{L} \qquad \rightarrow \qquad \lim_{L \to \pm\infty} = 0$$

Wobei L die betrachtete Weglänge darstellt. Das Teilchen hält sich unabhängig von seiner Geschwindigkeit überall und ständig mit gleicher Wahrscheinlichkeit auf. Der Ort ist unbestimmt. Daher gilt endgültig:

$$\phi = \sin\left(\frac{x}{\bar{h}} \cdot \sqrt{2m \cdot E}\right) \left[\frac{\mathrm{m}}{\mathrm{Js}}\sqrt{\mathrm{kgJ}} = \frac{\mathrm{m}}{\mathrm{Nms}}\sqrt{\mathrm{kgNm}} = \frac{\mathrm{s}}{\mathrm{kgm}}\sqrt{\frac{\mathrm{kg}^2\mathrm{m}^2}{\mathrm{s}^2}} = 1\right]$$

Wobei die Energie der freien Welle die kinetische Energie ist:

$$E = \frac{p^2}{2m}$$

$\Rightarrow$

$$\phi = \sin\left(x \cdot \sqrt{\frac{2m}{\bar{h}^2} \cdot \frac{p^2}{2m}}\right)$$

$\Rightarrow$

$$\phi(x) = \sin\left(\frac{p}{\bar{h}} \cdot x\right) \left[\frac{\mathrm{Ns}}{\mathrm{Js}}\mathrm{m} = \frac{\mathrm{N}}{\mathrm{Nm}}\mathrm{m} = 1\right]$$

Wobei p den Impuls des Teilchens darstellt.

$$\omega(x) = \frac{p}{\bar{h}} \left[\frac{\mathrm{Ns}}{\mathrm{Js}} = \frac{\mathrm{N}}{\mathrm{Nm}} = \frac{1}{\mathrm{m}}\right]$$

$\Rightarrow$

$$f_x = \frac{p}{h} \left[\frac{\mathrm{Ns}}{\mathrm{Js}} = \frac{\mathrm{N}}{\mathrm{Nm}} = \frac{1}{\mathrm{m}}\right]$$

Die de- Broglie- Relation:

$$p = \frac{h}{\lambda} = k \cdot \bar{h}$$

$\Rightarrow$

$$\phi = \sin\left(\frac{p}{\bar{h}} \cdot x\right) = \sin\left(\frac{2\pi}{\lambda} \cdot x\right)$$

Die Wellenlänge-Frequenz-Relation:

$$\lambda = \frac{c}{f_t}$$

$\Rightarrow$

$$\phi = \sin\left(\frac{2\pi}{c} \cdot f_t \cdot x\right)$$

Die Definition der Kreisfrequenz:

$$\omega = \omega_t = 2\pi \cdot f_t = 2\pi \cdot \frac{c}{\lambda} = \frac{c \cdot p}{\bar{h}}$$

$\Rightarrow$

$$\phi = \sin\left(\frac{\omega}{c} \cdot x\right)$$

Die Definition der nichtrelativistischen Geschwindigkeit:

$$c = \frac{x}{t}$$

$\Rightarrow$

$$\phi(t) = \sin(\omega t) \left[\frac{1}{\mathrm{s}}\mathrm{s} = 1\right]$$

Die Energie für eine Welle:

$$E = \bar{h}\omega \left[\mathrm{J}\frac{\mathrm{s}}{\mathrm{s}} = \mathrm{J}\right]$$

Mit:

$$\omega(t) = \frac{p}{\bar{h}} \cdot c \left[\frac{\mathrm{mNs}}{\mathrm{sJs}} = \frac{\mathrm{mN}}{\mathrm{sNm}} = \frac{1}{\mathrm{s}}\right]$$

Der Übergang zur zeitabhängigen Schrödinger-Gleichung erfolgt durch:

$$\psi(x;t) = \phi(x) \cdot e^{-\frac{i}{\bar{h}} \cdot E \cdot t}$$

$\Rightarrow$

$$\psi(x;t) = \phi(x) \cdot e^{-i\omega_t t}$$

$\Rightarrow$

$$\psi(x;t) = \sin(\omega_x x) \cdot e^{-i\omega_t t}$$

$\Rightarrow$

$$\psi(x;t) = \sin(\omega_x x)\left[\cos(\omega_t t) - i\sin(\omega_t t)\right]$$

$\Rightarrow$

$$\psi(x;t) = \sin(\omega_x x)\cos(\omega_t t) - i\sin(\omega_x x)\sin(\omega_t t)$$

Es erfolgt eine Zerlegung in Real- und Imaginäranteil:

$$\begin{aligned} \mathrm{Re}\ \psi &= +\sin(\omega_x x)\cos(\omega_t t) \\ \mathrm{Im}\ \psi &= -\sin(\omega_x x)\sin(\omega_t t) \end{aligned}$$

Die Ergebnisse werden den Achsen eines kartesischen Koordinatensystems zugeordnet und in polare umgewandelt:

$$\begin{aligned} x(\psi) &= \mathrm{Re}\ \psi \\ y(\psi) &= \mathrm{Im}\ \psi \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} x(\psi) &= +\sin(\omega_x x)\cos(\omega_t t) \\ y(\psi) &= -\sin(\omega_x x)\sin(\omega_t t) \end{aligned}$$

$\Rightarrow$

$$x(\psi)^2 + y(\psi)^2 = +\sin^2(\omega_x x)\left(\cos^2(\omega_t t) + \sin^2(\omega_t t)\right)$$
$$\frac{y}{x}(\psi) = -\frac{\sin(\omega_t t)}{\cos(\omega_t t)}$$

$\Rightarrow$

$$r = \sqrt{x(\psi)^2 + y(\psi)^2} = +\sin(\omega_x x)$$
$$\tan\varphi = \frac{y}{x}(\psi) = -\tan(\omega_t t)$$

$\Rightarrow$

$$r = +\sin(\omega_x x) \qquad \varphi = -\omega_t t$$

Es wurde berechnet:

$$\omega(x) = \frac{p}{\hbar} \qquad \omega(t) = \frac{p}{\hbar} \cdot c$$

$\Rightarrow$

$$\omega(t) = c \cdot \omega(x)$$

$\Rightarrow$

$$r = \sin(\omega_x x) \qquad \varphi = -c \cdot \omega_x t$$

Sowie:

$$c \cdot t = x$$

$\Rightarrow$

$$r(x) = \sin(\omega_x x) \qquad \varphi(x) = -\omega_x x$$

$\Rightarrow$

$$r(\varphi) = -\sin\varphi$$

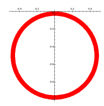
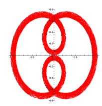
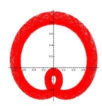
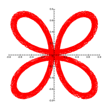
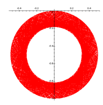
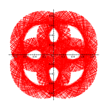
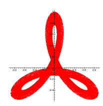
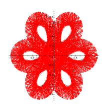
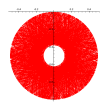
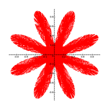
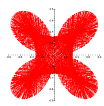
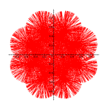
Oder man wählt folgende Umformung zwecks graphischer Veranschaulichung:

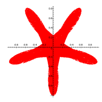
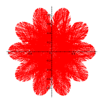
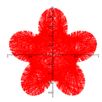
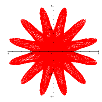
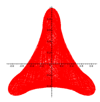
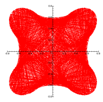
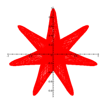
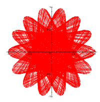
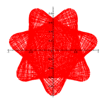
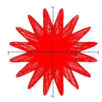
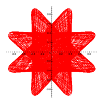
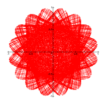
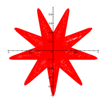
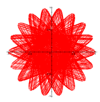
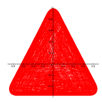
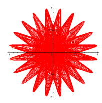
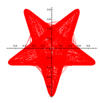
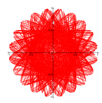
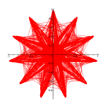
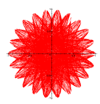
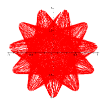
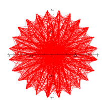
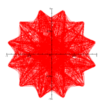
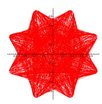
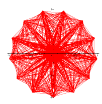
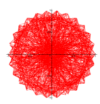
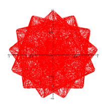
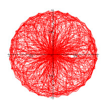
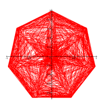
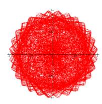
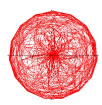
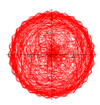
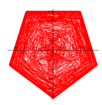
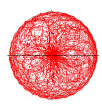
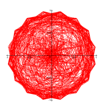
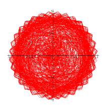
$$c \cdot t = x$$

$\Rightarrow$

$$r(t) = \sin(\omega_x c t) \qquad \varphi(t) = -\omega_t t$$

In Polarkoordinatendarstellung ergeben sich folgende Schaubilder:

$t = 0..1$	$\omega_t = 200\pi$	$\omega_t = 400\pi$	$\omega_t = 800\pi$
$\omega_x c = 200\pi$			
$\omega_x c = 400\pi$			
$\omega_x c = 600\pi$			
$\omega_x c = 800\pi$			

$\omega_x c = 1000\pi$			
$\omega_x c = 1200\pi$			
$\omega_x c = 1400\pi$			
$\omega_x c = 1600\pi$			
$\omega_x c = 1800\pi$			
$\omega_x c = 2000\pi$			
$\omega_x c = 2200\pi$			
$\omega_x c = 2400\pi$			
$\omega_x c = 2600\pi$			
$\omega_x c = 2800\pi$			
$\omega_x c = 3000\pi$			
$\omega_x c = 3200\pi$			

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